Logic and applications

Ramchandra Phawade
Department of Computer Science and Engineering
IIT Dharwad, India

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Substitution

Given α what is α_x^t?
Replace free occurrences of x by t in α.

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  - $\forall y \alpha$ if $y = x$,
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Given α what is α^t_x?
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- $\forall x \neg(\forall y(x = y)) \implies [\neg(\forall y(x = y))]^y_x$
  becomes
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- x is substitutable by t in $\forall y\alpha$ iff
  - Either x is not free in α
  - Or y does not occur in α and t is substitutable for x in α.

??

Proof system for FOL : A syntactic construct

- Axiom Groups and Axioms

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$$Axioms = I(\{Axg1, Axg2, Axg3, Axg4\}, \{\forall x, ...\}).$$

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$$\text{Assumption : } \alpha, \alpha \implies \beta$$
$$\text{Conclusion: } \beta$$

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- α has a proof iff $\alpha \in I(\{Axioms\}, \{MP\})$.

Example proof: $(\vdash P(x) \implies \exists x P(x))$

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(1) $\vdash (\forall x \neg P(x) \implies \neg P(x))$
Axg2 where $t = x$.

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(3) $\vdash (P(x) \implies \neg(\forall x \neg P(x))$
MP, (1),(2)

Soundness Theorem: $(\Gamma \vdash \alpha \implies \Gamma \models \alpha)$

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Proof by induction on structure of α.
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(b1) : $\alpha \in \Gamma$
Easy case.

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Easy case.

(b2) : $\alpha \in$ Axioms
Requires induction on structure of Axioms itself.

(b2-bc1) α is in Axg1
(b2-bc2) α is in Axg2
(b2-bc3) α is in Axg3
(b2-bc4) α is in Axg4
(b2-ind) $\alpha = \forall x \beta$, where β is an axiom.

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(ind) : α result of application of MP.

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(b2-bc2) α is in Axg2

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Proof by induction on structure of α.
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- ★ TP : $\models \forall x \alpha \implies \alpha^t_x$

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- $(M \not\models_s \forall x \alpha)$ because
  we have $M \not\models_{s(x|b)} \alpha$.

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- but we have $M \models_s \forall x\alpha$ which is a contradiction.

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$M \models_s \forall x\alpha$ iff for every $d \in \mathcal{U}^M$, we have $M \models_{s(x|d)} \alpha$.

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- $\bar{s}(t) = b$ where $b \in \mathcal{U}^M$.
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  we have $M \nvDash_{s(x|b)} \alpha$.
- but we have $M \vDash_s \forall x\alpha$ which is a contradiction.

$M \vDash_s \forall x\alpha$ iff for every $d \in \mathcal{U}^M$, we have $M \vDash_{s(x|d)} \alpha$.
What is this *new* function?

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What is this *new* function?
$s(x|d)(y) = s(y)$ if $y \neq x$,
$s(x|d)(y) = d$ if $y = x$.

Soundness Theorem: $(\Gamma \vdash \alpha \implies \Gamma \models \alpha)$

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(TP) $\Gamma \models (\forall x(\alpha \implies \beta) \implies (\forall x\alpha \implies \forall x\beta))$

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- Suppose not.
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  $M \models_s \Gamma$ and $M \models_s (\forall x(\alpha \implies \beta))$ and $(M \not\models_s (\forall x\alpha \implies \forall x\beta))$.

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- ★ So, $M \models_s \forall x\alpha$ and $M \not\models_s \forall x\beta$.

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- So, $M \models_s \forall x\alpha$ and $M \not\models_s \forall x\beta$.
- so we get an element b, for which β is false. i.e., $M \not\models_{s(x|b)} \beta$.

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- ★ So, $M \vDash_s \forall x\alpha$ and $M \nvDash_s \forall x\beta$.
- ★ so we get an element b, for which β is false. i.e., $M \nvDash_{s(x|b)} \beta$.
- ★ Since $M \vDash_s \forall x\alpha$, for element b, $M \vDash_{s(x|b)} \alpha$.

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  Then there exists a structure M and an assignment s such that $M \models_s \Gamma$ and $M \models_s (\forall x(\alpha \implies \beta))$ and $(M \not\models_s (\forall x\alpha \implies \forall x\beta))$.
- ★ So, $M \models_s \forall x\alpha$ and $M \not\models_s \forall x\beta$.
- ★ so we get an element b, for which β is false. i.e., $M \not\models_{s(x|b)} \beta$.
- ★ Since $M \models_s \forall x\alpha$, for element b, $M \models_{s(x|b)} \alpha$.
- ★ therefore, $\alpha \implies \beta$ is not true in M under s i.e., $(M \not\models_s \forall x(\alpha \implies \beta))$.

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- ★ So, $M \vDash_s \forall x\alpha$ and $M \nvDash_s \forall x\beta$.
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- ★ therefore, $\alpha \implies \beta$ is not true in M under s i.e., $(M \nvDash_s \forall x(\alpha \implies \beta))$.
- ★ which is a contradiction to $M \vDash_s (\forall x(\alpha \implies \beta))$

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Proof by induction on structure of α.
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(TP) $\Gamma \models \alpha \implies \forall x \alpha$, when $x \notin FV(\alpha)$.

Soundness Theorem: $(\Gamma \vdash \alpha \implies \Gamma \models \alpha)$

Proof by induction on structure of α.
$\alpha \in I(Axioms \cup \Gamma, \{MP\})$.

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(TP) $\Gamma \models \alpha \implies \forall x \alpha$, when $x \notin FV(\alpha)$.

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- ⋆ we get an element b in the universe, for which α is false. That is $M \nvDash_{s(x|b)} \alpha$.
- ⋆ but x in not free in α, therefore, $s(x|b)$ is same as s for any $y \neq x$.
  Hence, $M \nvDash_s \alpha$, which is a contradiction.

Soundness Theorem: ($\Gamma \vdash \alpha \implies \Gamma \vDash \alpha$)

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Proof by induction on structure of α.
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(ind) : α result of MP
routine (like PL).

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$M \vDash_s \exists x \alpha$ iff for some $d \in \mathcal{U}^M$, we have $M \vDash_{s(x|d)} \alpha$.
Where,
$s(x|d)(y) = s(y)$ if $y \neq x$,
$s(x|d)(y) = d$ if $y = x$.

Deduction Theorem for FOL

Theorem

For all Γ - a set of wffs, α and β,

$$\Gamma \cup \{\alpha\} \vdash \beta \textit{ iff } \Gamma \vdash (\alpha \implies \beta)$$

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Proof.

Exercise.

Hint: Same on the lines of proof of deduction theorem for PL. $\square$

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Theorem

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Then $\Gamma \vdash \forall x \alpha$.

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If Γ *- a set of wffs,* α *is a wff, such that* $x \notin FV(\Gamma)$*,*
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Rephrase the above statement as follows:
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So proof by structural induction on α.

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- (Base case 1) : $\alpha \in$ Axiom
  Then $\forall x \alpha \in$ Axioms, by structure of Axioms.

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  That is we have $\Gamma \vdash \beta$, $\Gamma \vdash \beta \implies \alpha$, then $\Gamma \vdash \alpha$.

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- $\Gamma \vdash \forall x(\beta \implies \alpha) \implies (\forall x \beta \implies \forall x \alpha)$ Axg3.

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- $\Gamma \vdash (\forall x \beta \implies \forall x \alpha)$ (MP).

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Example : TP: $\forall x \forall y R(x, y) \implies \forall y \forall x R(x, y)$

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6. $\forall x \forall y R(x, y) \vdash \forall x R(x, y)$ (GenThm as $x \notin FV(\Gamma)$).

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