Indian Institute of Technology Dharwad

Endsem: CS 621: Marking scheme and solutions

November 18, 2023

(Total marks : $8 * 10 = 80$)

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1. Suppose $L(x)$ means "x is a logician", $C(x)$ means "x is a celebrity" and $R(x)$ means "x is a rich", $S(x)$ means "x is a computer scientist".

   Translate the following sentences in predicate logic:

   (a) No logicians are celebrities.
   (b) Some celebrities are not logicians.
   (c) All rich logicians are computer scientists.
   (d) Not all logicians are computer scientists.
   (e) Some logicians are rich computer scientists.

   Marking scheme: 2 marks for each correct formulation.

2. Which of the following structures are models for the formula:

   $$\exists x \exists y \exists z (R(x,y) \wedge R(z,y) \wedge R(x,z) \wedge \neg R(z,x))$$

   (a) $\mathcal{M} = (\mathcal{U}^{\mathcal{M}} = \aleph, R^{\mathcal{M}} = \{(m,n) \mid m,n \in \aleph, m < n\})$
   (b) $\mathcal{M} = (\mathcal{U}^{\mathcal{M}} = \aleph, R^{\mathcal{M}} = \{(m,m+1) \mid m \in \aleph\})$
   (c) $\mathcal{M} = (\mathcal{U}^{\mathcal{M}} = 2^{\aleph}, R^{\mathcal{M}} = \{(A,B) \mid A,B \subseteq \aleph\}, A \subseteq B)$
   Note : Here, $\aleph$ denotes the set of natural numbers (it has 0 in it) and $2^{\aleph}$ denotes the powerset of natural numbers.

   Marking scheme: (*a*) 3 marks (*b*) 4 marks (*c*) 3 marks.

3. Prove or disprove the following:

   (a) $(\forall x \alpha \vee \forall x \beta) \vDash \forall x(\alpha \vee \beta)$ *This can be proved.*
   (b) $\forall x(\alpha \vee \beta) \vDash (\forall x \alpha \vee \forall x \beta)$ *Disprove this with a counterexample.*

   Marking scheme: 5 marks each.

4. Show that $\{\forall x \exists y \; (P(x) \vee Q(y))\} \vDash \exists y \forall x \; (P(x) \vee Q(y))$.

5. If $\Gamma \vdash \alpha_1, \ldots, \Gamma \vdash \alpha_n$ and $\{\alpha_1, \ldots, \alpha_n\} \vDash \beta$ then $\Gamma \vdash \beta$.

   Marking scheme: This proof requires repeated application of modus ponens. If you have not done this 3 marks are deducted. Marks are deducted for gaps in the proof.

6. Give a proof in the proof system for FOL discussed in the class:

   $$\forall x(F(x) \implies G(x)), \forall x(G(x) \implies H(x)) \vdash \forall x(F(x) \implies H(x))$$

   Marking scheme: A fully formal proof without any shortcuts or gaps gets 10 marks. Marks are cut if you do not support how a particular appears in the proof. For example if you use generalization theorem, but fail to write it, marks are deducted.

7. Consider the formula: $\phi = \alpha \wedge \beta \wedge \gamma$ where

$$\alpha = \forall x R(x, f(x)) \text{ and } \beta = \forall y \neg R(y, y)$$
$$\text{and } \gamma = \forall u \forall v \forall w ((R(u, v) \wedge R(v, w)) \implies R(u, w)).$$

Prove or disprove: (a) Formula ϕ is satisfiable. (b) ϕ has a finite model.

Model Ans(a): Set of natural numbers, with R as $<$ (*lessthan*) and f as successor relation.

Model Ans(b): It can not have a model of finite size. Take a finite model. Pick up an arbitrary element u of it, then build a sequence $u_0 = u, u_1, u_2, \ldots$ where $u_{i+1} = f(u_i)$.
Since the universe if finite there exists i and j such that $i < j$ and $u_i = u_j$. By α we get $(u_0, u_1) \in R, (u_0, u_1) \in R, (u_0, u_1) \in R, \ldots$. Therefore by γ we get $(u_i, u_j) \in R$. But we have $u_i = u_j$ so we have $(u_i, u_i) \in R$, which contradicts β.

Marking scheme: (a) 3 marks for showing that the formula is sat. (b) 7 marks for showing that it does not have a finite model. A formal proof is expected here to get full marks. Partial marks are given for the incomplete proofs. In this proof you can not assume any interpretation of R.

8. *Diameter of a graph*: Diameter of a graph is the longest path among all the shortest paths in the graph.

Prove the following claim: The family of graphs with finite diameters is not definable, in FOL with equality and one binary predicate.

Marking Scheme: Step (1) : Construct $\Sigma_{dia} = \{\phi_1, \phi_2, \ldots\}$ where each ϕ_i forces a path of length at least i, and hence a diameter of length at least i in the graph.

Step (2) Assuming that Σ^* exists, which defines all the finite diameter graphs, one has to show that $\Sigma_{dia} \cup \Sigma^*$ is not satisfiable.

Step (3) Show that each finite subset of $\Sigma_{dia} \cup \Sigma^*$ is satisfiable, and using Compactness theorem $\Sigma_{dia} \cup \Sigma^*$ is satisfiable, hence leading to contradiction.

Step (1) is for 3 marks; *Step* (2) carries 2 marks; and *Step* (3) has weight of 5 marks. One needs to invoke CT, if this is not done 2 marks are deducted.