CS621: Logic and applications

Ramchandra Phawade
Department of Computer Science and Engineering
IIT Dharwad, India

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Time slots

- 7A : Mon: 14:00 to 14:55
- 7B : Tue: 10:30 to 11:25
- 7C : Thu: 09:00 to 10:25

Evaluation Scheme

1. Assignments+Quiz : 30%
   (2 assignments + 1 Quiz with 10% weight for each)
   Out of two one is a programming assignment.
2. Midsem : 30%
3. Endsem : 40%

Textbooks and References

1. A mathematical introduction to logic
   Herbert B. Enderton
   Elsevier
2. Logic in Computer Science
   Authors: Huth and Ryan
   Cambridge University Press
3. Z3 tool
   SAT/SMT by example by Dennis Yurichev
   https://yurichev.com/SAT_SMT.html

Additional material:

- Logicomix : https://en.wikipedia.org/wiki/Logicomix
- Engines of Logic by Martin Davis

Why should one study this course?

before jumping into the answers/applications,
let us take a look at the history.

Gottfried Leibniz

Born : Leipzig, Germany ; 1646 Died : Hanover, Germany; 1716

Many contributions : philosophy, calculus, logic.

Gottfried Leibniz

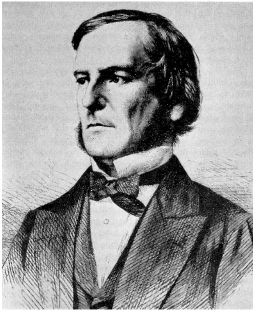
Believed : Human reasoning could be reduced to calculations.
The dream was – Let us compute. (build Machines)

How to represent human reasoning?

Logic! Symbols will have meanings. Create a system (algebra) to manipulate the symbols. Leibinz : calculus ratiocinator.

George Boole

Born : 1815, London; Died : 1864, Ireland.



Contributions:

- Boolean Logic – the basis of calculations in modern computer.
- Turns logic into algebra (Leibniz's dream !)
- Can not caputre all of human thoughts.

Gottlob Frege

Born : 1848, Germany; Died : 1925, Germany.

Contributions:

- Predicate Logic – the modern logic. Language of Mathematics.
- $\forall a, b, c, n \; [\; (a, b, c > 0 \wedge n > 2) \;\rightarrow\; a^n + b^n \neq c^n \;]$
- More powerful than boolean logic. But closer to Leibniz's dream.

Gottlob Frege

- Language of mathematics - predicate logic
- Developed axiomatization of set theory.
- Expressing set theory in terms of logic.

Georg Cantor

Born : 1845, Russia. Died : 1918, Germany.

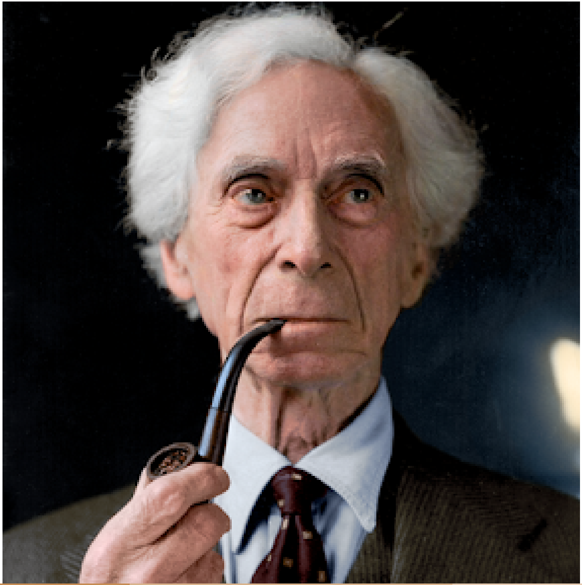
Georg Cantor

Contributions: Infinite sets, cardinality.

- Set of even numbers is of the same size of natural numbers.
- nonintuitive !
- Fierce opposition form Kronecker, Konig, Poincare, Weyl.

Bertrand Russell (1872-1970)

Born : 1872, England; Died : 1970, England



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  Therefore Barber belongs to S.
  Sets are not defined properly.

David Hilbert

Born : 1862, Könisberg   Died : 1943 : Göttingen, Germany

Program for securing foundations of Mathematics.

Securing foundations of Mathematics: David Hilbert

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- Decidability : There should be an algorithm to decide the truth of mathematical statements.

Principia Mathematica by Russell and Whitehead : One attempt in this direction.

Consistency of arithmetic: David Hilbert

- $\forall a, b, c, n \; [\; (a, b, c > 0 \wedge n > 2) \;] \; \rightarrow \; a^n + b^n \neq c^n$
  - Is there an finite and complete axiomatization of arithmetic which is consistent? (1920)

Kurt Gödel

Born : Brünn (now Czech Republic), 1906; Died : Princeton, 1978.

Major Contributions : Answer is NO!

Godel : Incompleteness Theorems

- First incompleteness theorem–arithmetic.
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- First incompleteness theorem–arithmetic.
  Any consistent formalism strong enough in which sufficint arithemetic can be carried out is not complete.
- Second incompleteness theorem: Any such formalism can not prove its own consistency.

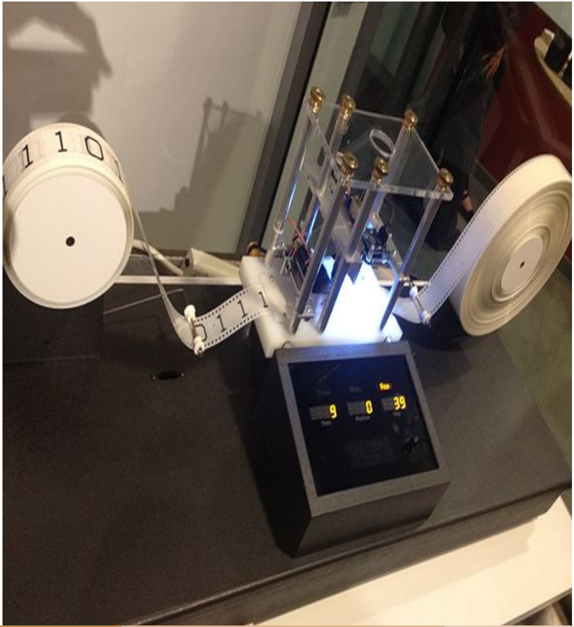
Entscheidungsproblem : David Hilbert

- $\forall a, b, c, n \; [\; (a, b, c > 0 \wedge n > 2) \;] \;\rightarrow\; a^n + b^n \neq c^n$
- Is there an "algorithm" that can take such a mathematical statement as input and say if it is true or false. (1900)

Mathematical notion of computation : Turing Machines

Figure: Alan Turing (1912-1954)

Turing machines



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  protocols – design, verification; AI.
- Automated verification of chips: LTL, CTL, automata theoretic approaches.

Formal Verification

- Systems – Automata, Different kinds of machines, programs,
- Property– specified by some suitable logic
- Does the system satisfy the given property?

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- Does the system satisfy the given property?
- Automated systems
  Is goal achievable? planning, vefification.
  verification of systems with machine learning components.
  2020 ACM doctoral thesis award.

What is in the course?

- formalization of proofs, theory, consistency, completeness, soundness, decidability
- propositional logic
- FOL – proof mechanism, undecidability, expressibility
- Decidable fragments–Presburger arithmetic
- Decision procedures for First Order Theories. SAT/SMT solvers.

Thank you