Logic and applications

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Proof system : A syntactic construct

Components of a proof system :

- Axioms
  - $(\alpha \implies (\beta \implies \alpha))$
  - $(\alpha \implies (\beta \implies \gamma)) \implies ((\alpha \implies \beta) \implies (\alpha \implies \gamma))$
  - $(\neg\alpha \implies \neg\beta) \implies (\beta \implies \alpha)$
- Deduction Rules : Modus Ponens

Assumption : $\alpha, \alpha \implies \beta$

Conclusion: β

Consistency

Definition (Consistency 1)

A set of wffs Σ is consistent if, there $\nexists\alpha$ such that $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$.

Definition (Consistency 2)

Σ is consistent iff $\exists\alpha$ such that $\Sigma \nvdash \alpha$.

consistency and satisfiability -01

Using Soundness theorem ($\Sigma \vdash \alpha \implies \Sigma \models \alpha$).

Theorem

Σ is satisfiable $\implies$ Σ is consistent.

- $\emptyset$ is consistent
- $\Sigma = \{p_i \implies p_j \mid p_i, i \in \mathbf{N}\}$ is consistent.
- Set of all wffs is not consistent

consistency and satisfiability -02

Theorem

Σ is consistent $\implies$ Σ is satisfiable .

consistency and satisfiability -02

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Σ is consistent $\implies$ Σ is satisfiable .

The rest of the lecture is about proving this theorem.

- Σ is consistent
- Σ' is Maximally consistent ($\Sigma \subseteq \Sigma'$)
- Prove that Σ' is satisfiable
- Σ is satisfiable.

Maximal consistency

A set of wffs Σ is maximally consistent if,

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- Maximality condition:
  For all α, either $\Sigma \vdash \alpha$ or $\Sigma \cup \{\alpha\}$ is not consistent.

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Come up with such an α.

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To prove that it is not maximally consistent, rule out both the cases in the second condition.
Come up with such an α.
Take $\alpha = p_3$.

- $\Sigma \nvdash \{p_3\}$.
  Otherwise we get a contradiction using soundness.
- $\{p_1, p_3\}$ is consistent as it is satisfiable.

Lemma

For any consistent set of wffs Σ*, there exists* Σ' *such that* $\Sigma \subseteq \Sigma'$ *and* Σ' *is maximally consistent.*

Lemma

For any consistent set of wffs Σ, there exists Σ' such that $\Sigma \subseteq \Sigma'$ and Σ' is maximally consistent.

- Enumerate all wffs $\alpha_1, \alpha_2, \ldots$

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- Define $\Sigma_0 = \Sigma$

Lemma

For any consistent set of wffs Σ, there exists Σ' such that $\Sigma \subseteq \Sigma'$ and Σ' is maximally consistent.

- Enumerate all wffs $\alpha_1, \alpha_2, \ldots$
- Define $\Sigma_0 = \Sigma$
- Define

$$\Sigma_{i+1} = \begin{cases} \Sigma_i & \text{if } \ \Sigma_i \vdash \neg(\alpha_{i+1}) \\ \Sigma_i \cup \{\alpha_{i+1}\} & \text{otherwise.} \end{cases}$$

Lemma

For any consistent set of wffs Σ, there exists Σ' such that $\Sigma \subseteq \Sigma'$ and Σ' is maximally consistent.

- Enumerate all wffs $\alpha_1, \alpha_2, \ldots$
- Define $\Sigma_0 = \Sigma$
- Define
$$\Sigma_{i+1} = \begin{cases} \Sigma_i & \text{if } \Sigma_i \vdash \neg(\alpha_{i+1}) \\ \Sigma_i \cup \{\alpha_{i+1}\} & \text{otherwise.} \end{cases}$$
- Define $\Sigma' = \bigcup_{i \in \mathbf{N}}^{\infty} \Sigma_i$

Now we prove that Σ' is maximally consistent.

Properties of the sets used to construct Σ' : (Step 1)

$$\Sigma_{i+1} = \begin{cases} \Sigma_i & \text{if } \Sigma_i \vdash \neg(\alpha_{i+1}) \\ \Sigma_i \cup \{\alpha_{i+1}\} & \text{otherwise.} \end{cases}$$

1. each Σ_{i+1} is consistent
2. for every i (index of enumeration of wffs)
   Either $\Sigma_{i+1} \vdash \alpha_{i+1}$ or $\Sigma_{i+1} \vdash \neg(\alpha_{i+1})$

Proving condition (2) is very easy.
Proving condition (1) is little bit tedious.
Assume that each Σ_i satisfies both these conditions, and prove that Σ' is Maximally consistent.

Proving that Σ_{i+1} is consistent

$$\Sigma_{i+1} = \begin{cases} \Sigma_i & \text{if } \Sigma_i \vdash \neg(\alpha_{i+1}) \\ \Sigma_i \cup \{\alpha_{i+1}\} & \text{otherwise.} \end{cases}$$

- Case 1: $\Sigma_{i+1} = \Sigma_i$.
  by IH we know that Σ_i is consistent.
- Case 2: $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_{i+1}\}$.

Proof: Case 2: $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_{i+1}\}$ is consistent

Proof by contradiction.

1. $\Sigma_i \cup \{\alpha_{i+1}\}$ is not consistent

Proof: Case 2: $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_{i+1}\}$ is consistent

Proof by contradiction.

1. $\Sigma_i \cup \{\alpha_{i+1}\}$ is not consistent
2. $\Sigma_i \cup \{\alpha_{i+1}\} \vdash \beta$

Proof: Case 2: $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_{i+1}\}$ is consistent

Proof by contradiction.

1. $\Sigma_i \cup \{\alpha_{i+1}\}$ is not consistent
2. $\Sigma_i \cup \{\alpha_{i+1}\} \vdash \beta$
3. In particular $\beta = \neg(\alpha_{i+1})$, so
   $\Sigma_i \cup \{\alpha_{i+1}\} \vdash \neg(\alpha_{i+1})$

Proof: Case 2: $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_{i+1}\}$ is consistent

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3. In particular $\beta = \neg(\alpha_{i+1})$, so
   $\Sigma_i \cup \{\alpha_{i+1}\} \vdash \neg(\alpha_{i+1})$
4. by Deduction theorem
   $\Sigma_i \vdash (\alpha_{i+1} \implies \neg(\alpha_{i+1}))$

Proof: Case 2: $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_{i+1}\}$ is consistent

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2. $\Sigma_i \cup \{\alpha_{i+1}\} \vdash \beta$
3. In particular $\beta = \neg(\alpha_{i+1})$, so $\Sigma_i \cup \{\alpha_{i+1}\} \vdash \neg(\alpha_{i+1})$
4. by Deduction theorem $\Sigma_i \vdash (\alpha_{i+1} \implies \neg(\alpha_{i+1}))$
5. Use $\vdash (\gamma \implies \neg\gamma) \implies \neg\gamma$ (Exercise)

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   $\Sigma_i \vdash (\alpha_{i+1} \implies \neg(\alpha_{i+1}))$
5. Use $\vdash (\gamma \implies \neg\gamma) \implies \neg\gamma$ (Exercise)
6. put $\gamma = \alpha_{i+1}$ to get
   $\Sigma_i \vdash (\alpha_{i+1} \implies \neg(\alpha_{i+1})) \implies \neg(\alpha_{i+1})$

Proof: Case 2: $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_{i+1}\}$ is consistent

Proof by contradiction.

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6. put $\gamma = \alpha_{i+1}$ to get
   $\Sigma_i \vdash (\alpha_{i+1} \implies \neg(\alpha_{i+1})) \implies \neg(\alpha_{i+1})$
7. $\Sigma_i \vdash \neg(\alpha_{i+1})$ MP, 4, 6

Proof: Case 2: $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_{i+1}\}$ is consistent

Proof by contradiction.

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2. $\Sigma_i \cup \{\alpha_{i+1}\} \vdash \beta$
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7. $\Sigma_i \vdash \neg(\alpha_{i+1})$ MP, 4, 6
8. but then $\Sigma_{i+1} = \Sigma_i$, a contradiction.

Prove that Σ' is maximally consistent

- Prove that Σ' is consistent
- Prove that Σ' satisfies the maximality condition :
  For all α, either $\Sigma \vdash \alpha$ or $\Sigma \cup \{\alpha\}$ is not consistent.

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- For some wff α, $\Sigma' \vdash \alpha$ and $\Sigma' \vdash \neg\alpha$.

Claim 1: Prove that Σ' is consistent

Proof by contradiction.

- Assume that Σ' is inconsistent.
- For some wff α, $\Sigma' \vdash \alpha$ and $\Sigma' \vdash \neg\alpha$.
- Proof are finite.

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Proof by contradiction.

- Assume that Σ' is inconsistent.
- For some wff α, $\quad \Sigma' \vdash \alpha$ and $\Sigma' \vdash \neg\alpha$.
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- Assume that Σ' is inconsistent.
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- All assumptions used in $\Sigma' \vdash \alpha$ belong to some Σ_i.
- All assumptions used in $\Sigma' \vdash \neg\alpha$ belong to some Σ_j.
- $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \subseteq \Sigma_i \ldots \subseteq \Sigma_j \ldots \subseteq \Sigma_k \ldots$

Claim 1: Prove that Σ' is consistent

Proof by contradiction.

- Assume that Σ' is inconsistent.
- For some wff α, $\Sigma' \vdash \alpha$ and $\Sigma' \vdash \neg\alpha$.
- Proof are finite.
- All assumptions used in $\Sigma' \vdash \alpha$ belong to some Σ_i.
- All assumptions used in $\Sigma' \vdash \neg\alpha$ belong to some Σ_j.
- $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \subseteq \Sigma_i \ldots \subseteq \Sigma_j \ldots \subseteq \Sigma_k \ldots$
- $\Sigma_k \vdash \alpha$ and $\Sigma_k \vdash \neg\alpha$

Claim 1: Prove that Σ' is consistent

Proof by contradiction.

- Assume that Σ' is inconsistent.
- For some wff α, $\Sigma' \vdash \alpha$ and $\Sigma' \vdash \neg\alpha$.
- Proof are finite.
- All assumptions used in $\Sigma' \vdash \alpha$ belong to some Σ_i.
- All assumptions used in $\Sigma' \vdash \neg\alpha$ belong to some Σ_j.
- $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \subseteq \Sigma_i \ldots \subseteq \Sigma_j \ldots \subseteq \Sigma_k \ldots$
- $\Sigma_k \vdash \alpha$ and $\Sigma_k \vdash \neg\alpha$
- A contradiction, as Σ_k is inconsistent now.

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  $\Sigma_i \vdash \alpha_i$ or $\Sigma_i \vdash \neg\alpha_i$
  i.e., $\Sigma_i \vdash \alpha$ or $\Sigma_i \vdash \neg\alpha$
- $\Sigma' \vdash \alpha$ or $\Sigma' \vdash \neg\alpha$

Proving that Σ' is satisfiable

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Define a valuation function as follows:

$$V_{\Sigma'}(p) = \begin{cases} \mathit{True}, & \text{if } \Sigma' \vdash p \\ \mathit{False}, & \text{otherwise.} \end{cases}$$

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Claim (Sufficient to prove that Σ' is satisfiable)

For all α, $\Sigma' \vdash \alpha$ iff $V_{\Sigma'}(\alpha) = \mathit{True}$

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \textit{True}$

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- Base case: $\gamma = p$
  $\Sigma' \vdash p$ iff $V_{\Sigma'}(p) = True$.
  By definition of valuation function.

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- Induction step (1) : $\gamma = (\neg\alpha)$

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- Base case: $\gamma = p$
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  By definition of valuation function.
- Induction step (1) : $\gamma = (\neg\alpha)$
- Induction step (2) : $\gamma = (\alpha \implies \beta)$

We use adequacy of set $\{\neg, \implies\}$ for the propositional logic.

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Induction step (1) : $\gamma = (\neg\alpha)$

- Assume : $\Sigma' \vdash (\neg\alpha)$
- iff $\Sigma' \nvdash \alpha$ (consistency of Σ')
- iff $V_{\Sigma'}(\alpha) = \mathit{False}$ (Induction Hypothesis)
- iff $V_{\Sigma'}(\neg\alpha) = \mathit{True}$ ($\neg$ truth table)

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = True$

Induction step (1) : $\gamma = (\neg\alpha)$

- Assume : $\Sigma' \vdash (\neg\alpha)$
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- iff $V_{\Sigma'}(\alpha) = False$ (Induction Hypothesis)
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- Assume : $\Sigma' \nvdash (\neg\alpha)$
- iff $\Sigma' \vdash \alpha$ (maximality of Σ')
- iff $V_{\Sigma'}(\alpha) = True$ (Induction Hypothesis)
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Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = True$

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- Assume : $\Sigma' \nvdash (\neg\alpha)$
- iff $\Sigma' \vdash \alpha$ (maximality of Σ')
- iff $V_{\Sigma'}(\alpha) = True$ (Induction Hypothesis)
- iff $V_{\Sigma'}(\neg\alpha) = False$ ($\neg$ truth table)

(contrapositive statement of reverse direction is proved).
See that both directions of the claim are proved simultaneously so far.

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \vdash (\alpha \implies \beta)$ one direction of claim

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \vdash (\alpha \implies \beta)$ one direction of claim

- by Maximality of Σ', either $\Sigma' \vdash \alpha$ or $\Sigma' \vdash (\neg\alpha)$

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- by Maximality of Σ', either $\Sigma' \vdash \alpha$ or $\Sigma' \vdash (\neg\alpha)$
- In both cases we have to prove that $V_{\Sigma'}(\gamma) = True$.

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Case (1) : $\Sigma' \vdash \alpha$

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

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Case (1) : $\Sigma' \vdash \alpha$

- $\Sigma' \vdash \alpha$ (Assumption)

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \vdash (\alpha \implies \beta)$ one direction of claim

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- $\Sigma' \vdash \alpha$ (Assumption)
- $\Sigma' \vdash (\alpha \implies \beta)$ (Assumption)

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Case (1) : $\Sigma' \vdash \alpha$

- $\Sigma' \vdash \alpha$ (Assumption)
- $\Sigma' \vdash (\alpha \implies \beta)$ (Assumption)
- $\Sigma' \vdash \beta$ (MP 1,2)

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = True$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \vdash (\alpha \implies \beta)$ one direction of claim

- by Maximality of Σ', either $\Sigma' \vdash \alpha$ or $\Sigma' \vdash (\neg\alpha)$
- In both cases we have to prove that $V_{\Sigma'}(\gamma) = True$.

Case (1) : $\Sigma' \vdash \alpha$

- $\Sigma' \vdash \alpha$ (Assumption)
- $\Sigma' \vdash (\alpha \implies \beta)$ (Assumption)
- $\Sigma' \vdash \beta$ (MP 1,2)
- Now
- $\Sigma' \vdash \alpha$ iff $V_{\Sigma'}(\alpha) = True$ (IH)

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- $\Sigma' \vdash \beta$ iff $V_{\Sigma'}(\beta) = True$ (IH)

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

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Case (1) : $\Sigma' \vdash \alpha$

- $\Sigma' \vdash \alpha$ (Assumption)
- $\Sigma' \vdash (\alpha \implies \beta)$ (Assumption)
- $\Sigma' \vdash \beta$ (MP 1,2)
- Now
- $\Sigma' \vdash \alpha$ iff $V_{\Sigma'}(\alpha) = \mathit{True}$ (IH)
- $\Sigma' \vdash \beta$ iff $V_{\Sigma'}(\beta) = \mathit{True}$ (IH)
- iff $V_{\Sigma'}(\alpha \implies \beta) = \mathit{True}$ (truth table of $\implies$)

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = True$

$$\text{Induction step (2)} : \gamma = (\alpha \implies \beta)$$

Assume : $\Sigma' \vdash (\alpha \implies \beta)$ one direction of claim

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

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- by Maximality of Σ', either $\Sigma' \vdash \alpha$ or $\Sigma' \vdash (\neg\alpha)$
- In both cases we have to prove that $V_{\Sigma'}(\gamma) = \mathit{True}$.

Case (2) : $\Sigma' \vdash \neg\alpha$

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = True$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \vdash (\alpha \implies \beta)$ one direction of claim

- by Maximality of Σ', either $\Sigma' \vdash \alpha$ or $\Sigma' \vdash (\neg\alpha)$
- In both cases we have to prove that $V_{\Sigma'}(\gamma) = True$.

Case (2) : $\Sigma' \vdash \neg\alpha$

- $\Sigma' \nvdash \alpha$ (Consistency of Σ')

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \textit{True}$

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- iff $V_{\Sigma'}(\alpha) = \textit{False}$ (by IH, as α is subformula of γ)

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Case (2) : $\Sigma' \vdash \neg\alpha$

- $\Sigma' \nvdash \alpha$ (Consistency of Σ')
- iff $V_{\Sigma'}(\alpha) = \mathit{False}$ (by IH, as α is subformula of γ)
- iff $V_{\Sigma'}(\alpha \implies \beta) = \mathit{True}$ (truth table of $\implies$)

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \nvdash (\alpha \implies \beta)$ other direction of claim : contrapositive

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \nvdash (\alpha \implies \beta)$ other direction of claim : contrapositive

- We have $\Sigma' \nvdash \beta$

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \nvdash (\alpha \implies \beta)$ other direction of claim : contrapositive

- We have $\Sigma' \nvdash \beta$
  - Otherwise we have $\Sigma' \vdash \beta$ (maximality)

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \nvdash (\alpha \implies \beta)$ other direction of claim : contrapositive

- We have $\Sigma' \nvdash \beta$
  - Otherwise we have $\Sigma' \vdash \beta$ (maximality)
  - $\Sigma' \vdash (\beta \implies (\alpha \implies \beta))$ (Axiom 1)

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \nvdash (\alpha \implies \beta)$ other direction of claim : contrapositive

- We have $\Sigma' \nvdash \beta$
  - Otherwise we have $\Sigma' \vdash \beta$ (maximality)
  - $\Sigma' \vdash (\beta \implies (\alpha \implies \beta))$ (Axiom 1)
  - $\Sigma' \vdash (\alpha \implies \beta)$ (contradiction) (MP)

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \nvdash (\alpha \implies \beta)$ other direction of claim : contrapositive

- We have $\Sigma' \nvdash \beta$
  - Otherwise we have $\Sigma' \vdash \beta$ (maximality)
  - $\Sigma' \vdash (\beta \implies (\alpha \implies \beta))$ (Axiom 1)
  - $\Sigma' \vdash (\alpha \implies \beta)$ (contradiction) (MP)
- iff $V_{\Sigma'}(\beta) = \mathit{False}$ (by IH)

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \nvdash (\alpha \implies \beta)$ other direction of claim : contrapositive

- We have $\Sigma' \nvdash \beta$
  - Otherwise we have $\Sigma' \vdash \beta$ (maximality)
  - $\Sigma' \vdash (\beta \implies (\alpha \implies \beta))$ (Axiom 1)
  - $\Sigma' \vdash (\alpha \implies \beta)$ (contradiction) (MP)
- iff $V_{\Sigma'}(\beta) = \mathit{False}$ (by IH)
- $V_{\Sigma'}(\alpha \implies \beta) = \mathit{False}$ unless $V_{\Sigma'}(\alpha) = \mathit{False}$.

Proof of Claim: For all γ, $\Sigma' \vdash \gamma$ iff $V_{\Sigma'}(\gamma) = \mathit{True}$

Induction step (2) : $\gamma = (\alpha \implies \beta)$

Assume : $\Sigma' \nvdash (\alpha \implies \beta)$ other direction of claim : contrapositive

- We have $\Sigma' \nvdash \beta$
  - Otherwise we have $\Sigma' \vdash \beta$ (maximality)
  - $\Sigma' \vdash (\beta \implies (\alpha \implies \beta))$ (Axiom 1)
  - $\Sigma' \vdash (\alpha \implies \beta)$ (contradiction) (MP)
- iff $V_{\Sigma'}(\beta) = \mathit{False}$ (by IH)
- $V_{\Sigma'}(\alpha \implies \beta) = \mathit{False}$ unless $V_{\Sigma'}(\alpha) = \mathit{False}$.
- We rule out the case $V_{\Sigma'}(\alpha) = \mathit{False}$.

ruling out the case $V_\Sigma(\alpha) = \mathit{False}$.

ruling out the case $V_{\Sigma}(\alpha) = \mathit{False}$.

- Assume : $V_{\Sigma}(\alpha) = \mathit{False}$.

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- Assume : $V_\Sigma(\alpha) = \mathit{False}$.
- iff $\Sigma' \nvdash \alpha$ (by IH).

ruling out the case $V_\Sigma(\alpha) = \mathit{False}$.

- Assume : $V_\Sigma(\alpha) = \mathit{False}$.
- iff $\Sigma' \nvdash \alpha$ (by IH).
- $\Sigma' \vdash \neg\alpha$ (by maximality).

ruling out the case $V_\Sigma(\alpha) = \textit{False}$.

- Assume : $V_\Sigma(\alpha) = \textit{False}$.
- iff $\Sigma' \nvdash \alpha$ (by IH).
- $\Sigma' \vdash \neg\alpha$ (by maximality).
- $\Sigma' \cup \{\alpha\} \cup \{\neg\alpha\} \vdash \beta$ (LHS is inconsistent)

ruling out the case $V_\Sigma(\alpha) = \mathit{False}$.

- Assume : $V_\Sigma(\alpha) = \mathit{False}$.
- iff $\Sigma' \nvdash \alpha$ (by IH).
- $\Sigma' \vdash \neg\alpha$ (by maximality).
- $\Sigma' \cup \{\alpha\} \cup \{\neg\alpha\} \vdash \beta$ (LHS is inconsistent)
- $\Sigma' \cup \{\neg\alpha\} \vdash (\alpha \implies \beta)$ (deduction theorem)

ruling out the case $V_\Sigma(\alpha) = False$.

- Assume : $V_\Sigma(\alpha) = False$.
- iff $\Sigma' \nvdash \alpha$ (by IH).
- $\Sigma' \vdash \neg\alpha$ (by maximality).
- $\Sigma' \cup \{\alpha\} \cup \{\neg\alpha\} \vdash \beta$ (LHS is inconsistent)
- $\Sigma' \cup \{\neg\alpha\} \vdash (\alpha \implies \beta)$ (deduction theorem)
- $\Sigma' \vdash (\alpha \implies \beta)$ (contradiction) (Since $\Sigma' \vdash \neg\alpha$ and monotonicity)

consistency and satisfiability

Theorem

Σ is satisfiable $\implies$ Σ is consistent.

Theorem

Σ is consistent $\implies$ Σ is satisfiable .

- Σ is consistent
- Σ' is Maximally consistent ($\Sigma \subseteq \Sigma'$)
- Prove that Σ' is satisfiable
- Σ is satisfiable.

Completeness theorem : $\Sigma \models \alpha \implies \Sigma \vdash \alpha$

Proof by contradiction:

- Assumption : $\Sigma \nvdash \alpha$
- iff $\Sigma \cup \{\neg\alpha\}$ is consistent
- iff $\Sigma \cup \{\neg\alpha\}$ is satisfiable (Last result).
- iff $\Sigma \not\models \alpha$ (definition of $\models$).

Applications

- compactness theorem
- maximal satisfiability iff maximal consistency

Compactness theorem : one more proof

Σ is satifiable iff $\forall A \subseteq_{fin} \Sigma$ is satisfiable.

- Σ is satisfiable
- iff Σ is consistent (Thm: satisfiablity $\implies$ consistency)
- iff $A \subseteq_{fin} \Sigma$ is consistent (inconsistency is due to finite subsets)
- iff $A \subseteq_{fin} \Sigma$ is satisfiable (Thm: satisfiablity $\Leftarrow$ consistency)

Thm: Σ maximally satisfiable iff Σ maximally consistent

Σ maximally satisfiable $\implies$ Σ maximally consistent

- Assumption: Σ maximally satisfiable
- Σ is satisfiable
- Σ is consistent (by Thm proved earlier -1).
- $\forall\alpha$, either $\Sigma \models \alpha$ or $\Sigma \models \neg\alpha$ (maximal satisfiability)
- $\forall\alpha$, either $\Sigma \vdash \alpha$ or $\Sigma \vdash \neg\alpha$ (by completeness theorem)
- Σ is maximally consistent, as required.

Thm: Σ maximally satisfiable iff Σ maximally consistent

Σ maximally satisfiable $\Leftarrow$ Σ maximally consistent

- Assumption: Σ maximally consistent
- Σ is consistent
- Σ is satisfiable (by Thm proved earlier -1).
- $\forall \alpha$, either $\Sigma \vdash \alpha$ or $\Sigma \vdash \neg\alpha$ (maximal consistency)
- $\forall \alpha$, either $\Sigma \models \alpha$ or $\Sigma \models \neg\alpha$ (by soundness theorem)
- Σ is maximally satisfiable, as required.