Fourier Transforms

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1 Discrete vs. continuous FT

continuous Fourier transform

$$\hat{f}(k) = \int\limits_{-\pi}^{\pi} f(x) \cdot e^{-2\pi i k x} \mathrm{d}x \tag{1}$$

discrete Fourier transform

$$\hat{f}_k = \frac{1}{d} \sum_{n=0}^{N-1} f\left(x = \frac{n}{N} d\right) \cdot e^{\left(-2\pi i \cdot \frac{k}{d} \cdot \frac{n}{N} \cdot d\right)} \cdot \frac{d}{N} \tag{2}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} f\left(x = \frac{n}{N} d\right) \cdot e^{-\frac{2\pi i}{N} \cdot k \cdot n} \tag{3}$$

2 Differentation

Expression in frequency and angular frequency

$$\hat{f}(k) = \frac{1}{d} \int\limits_{0}^{d} f(x) e^{\frac{-2\pi i}{d} k x} \tag{4}$$

$$= \frac{1}{d} \int\limits_{0}^{d} f(x) e^{-2\pi i \xi x} \mathrm{d}x \tag{5}$$

$$\hat{f}' = \frac{\partial}{\partial x}\left(\int_{-\infty}^{\infty} \hat{f}(x) \cdot e^{i\xi x} \mathrm{d}k\right) \tag{6}$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\hat{f}(x) \cdot e^{i\xi x}\right) \mathrm{d}k \tag{7}$$

$$= \int_{-\infty}^{\infty} i\xi \cdot \hat{f}(x) \cdot e^{i\xi x} \mathrm{d}k \tag{8}$$

3 Transform

example with $N = 4$ and $x = \sin\left(\frac{n}{N} \cdot 2\pi\right)$

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{2\pi i}{N} \cdot k \cdot n}; \quad k = 0; 1; ..; N-1 \tag{9}$$

$$= \sum_{n=0}^{N-1} x_n \cdot \left(\cos\left(-\frac{2\pi}{N} \cdot k \cdot n\right) + i \cdot \sin\left(-\frac{2\pi}{N} \cdot k \cdot n\right)\right) \tag{10}$$

$$X_0 = \sum_{n=0}^{N-1} x_n e \tag{11}$$

$$= 0 + 1 + 0 + (-1) = 0 \tag{12}$$

$$X_1 = \sum_{n=0}^{N-1} x_n e^{-i\frac{2\pi}{N} \cdot 1 \cdot n} = 0 + e^{-i\frac{2\pi}{N} \cdot 1 \cdot 1} + 0 + e^{-i\frac{2\pi}{N} \cdot 1 \cdot 3} \tag{13}$$

$$= 0 + (-2i) \tag{14}$$

$$X_2 = 0 + 0i \tag{15}$$

$$X_3 = 0 + 2i \tag{16}$$

X_2 is Nyquist frequency and has only a real part, X_3 is conjugate complex of X_1 for real only input.

4 Inverse Transform

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{\frac{2\pi i}{N} \cdot k \cdot n} \tag{17}$$

$$x_0 = \frac{1}{4} \left(0 - 2ie^0 + 0 + 2ie^0\right) = 0 \tag{18}$$

$$x_1 = \frac{1}{4} \left(0 - 2ie^{\frac{2\pi i}{4} \cdot 1 \cdot 1} + 0 + 2ie^{\frac{2\pi i}{4} \cdot 3 \cdot 1}\right) = 1 \tag{19}$$

$$x_2 = 0 \tag{20}$$

$$x_1 = -1 \tag{21}$$