

EE 261 The Fourier Transform and its Applications

This Being an Ancient Formula Sheet
Handed Down
To All EE 261 Students

Integration by parts:

$$\int_a^b u(t)v'(t) dt = \left[u(t)v(t) \right]_{t=a}^{t=b} - \int_a^b u'(t)v(t) dt$$

Even and odd parts of a function: Any function $f(x)$ can be written as

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

(even part) (odd part)

Geometric series:

$$\sum_{n=0}^N r^n = \frac{1 - r^{N+1}}{1 - r}$$

$$\sum_{n=M}^N r^n = \frac{r^M(1 - r^{N-M+1})}{(1 - r)}$$

Complex numbers: $z = x + iy$, $\bar{z} = x - iy$, $|z|^2 = z\bar{z} = x^2 + y^2$

$$\frac{1}{i} = -i$$

$$x = \operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad y = \operatorname{Im} z = \frac{z - \bar{z}}{2i}$$

Complex exponentials:

$$e^{2\pi it} = \cos 2\pi t + i \sin 2\pi t$$

$$\cos 2\pi t = \frac{e^{2\pi it} + e^{-2\pi it}}{2}, \quad \sin 2\pi t = \frac{e^{2\pi it} - e^{-2\pi it}}{2i}$$

Polar form:

$$z = x + iy \quad z = re^{i\theta}, \quad r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}(y/x)$$

Symmetric sum of complex exponentials (special case of geometric series):

$$\sum_{n=-N}^N e^{2\pi int} = \frac{\sin(2N+1)\pi t}{\sin \pi t}$$

Fourier series If $f(t)$ is periodic with period T its Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T}$$

$$c_n = \frac{1}{T} \int_0^T e^{-2\pi int/T} f(t) dt = \frac{1}{T} \int_{-T/2}^{T/2} e^{-2\pi int/T} f(t) dt$$

Orthogonality of the complex exponentials:

$$\int_0^T e^{2\pi int/T} e^{-2\pi imt/T} dt = \begin{cases} 0, & n \neq m \\ T, & n = m \end{cases}$$

The normalized exponentials $(1/\sqrt{T})e^{2\pi int/T}$, $n = 0, \pm 1, \pm 2, \dots$ form an orthonormal basis for $L^2([0, T])$

Rayleigh (Parseval): If $f(t)$ is periodic of period T then

$$\frac{1}{T} \int_0^T |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

The Fourier Transform:

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi isx} dx$$

The Inverse Fourier Transform:

$$\mathcal{F}^{-1}f(x) = \int_{-\infty}^{\infty} f(s)e^{2\pi isx} ds$$

Symmetry & Duality Properties:

Let $f^-(x) = f(-x)$.

- $\mathcal{F}\mathcal{F}f = f^-$
- $\mathcal{F}^{-1}f = \mathcal{F}f^-$
- $\mathcal{F}f^- = (\mathcal{F}f)^-$
- If f is even (odd) then $\mathcal{F}f$ is even (odd)
- If f is real valued, then $\overline{\mathcal{F}f} = (\mathcal{F}f)^-$

Convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y) dy$$

- $f * g = g * f$
- $(f * g) * h = (f * g) * h$
- $f * (g + h) = f * g + f * h$

Smoothing: If f (or g) is p -times continuously differentiable, $p \geq 0$, then so is $f * g$ and

$$\frac{d^k}{dx^k}(f * g) = \left(\frac{d^k}{dx^k}f\right) * g$$

Convolution Theorem:

$$\mathcal{F}(f * g) = (\mathcal{F}f)(\mathcal{F}g)$$

$$\mathcal{F}(fg) = \mathcal{F}f * \mathcal{F}g$$

Autocorrelation: Let $g(x)$ be a function satisfying $\int_{-\infty}^{\infty} |g(x)|^2 dx < \infty$ (finite energy) then

$$\begin{aligned} (\bar{g} \star g)(x) &= \int_{-\infty}^{\infty} g(y) \overline{g(y-x)} dy \\ &= g(x) \star \overline{g(-x)} \end{aligned}$$

Cross correlation: Let $g(x)$ and $h(x)$ be functions with finite energy. Then

$$\begin{aligned} (\bar{g} \star h)(x) &= \int_{-\infty}^{\infty} \overline{g(y)} h(y+x) dy \\ &= \int_{-\infty}^{\infty} \overline{g(y-x)} h(y) dy \\ &= \overline{(\bar{h} \star g)(-x)} \end{aligned}$$

Rectangle and triangle functions

$$\Pi(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & |x| \geq \frac{1}{2} \end{cases} \quad \Lambda(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| \geq 1 \end{cases}$$

$$\mathcal{F}\Pi(s) = \text{sinc } s = \frac{\sin \pi s}{\pi s}, \quad \mathcal{F}\Lambda(s) = \text{sinc}^2 s$$

Scaled rectangle function

$$\Pi_p(x) = \Pi(x/p) = \begin{cases} 1, & |x| < \frac{p}{2} \\ 0, & |x| \geq \frac{p}{2} \end{cases},$$

$$\mathcal{F}\Pi_p(s) = p \text{sinc } ps$$

Scaled triangle function

$$\Lambda_p(x) = \Lambda(x/p) = \begin{cases} 1 - |x/p|, & |x| \leq p \\ 0, & |x| \geq p \end{cases},$$

$$\mathcal{F}\Lambda_p(s) = p \text{sinc}^2 ps$$

Gaussian

$$\mathcal{F}(e^{-\pi t^2}) = e^{-\pi s^2}$$

Gaussian with variance σ^2

$$g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \quad \mathcal{F}g(s) = e^{-2\pi^2\sigma^2 s^2}$$

One-sided exponential decay

$$f(t) = \begin{cases} 0, & t < 0, \\ e^{-at}, & t \geq 0. \end{cases} \quad \mathcal{F}f(s) = \frac{1}{a + 2\pi is}$$

Two-sided exponential decay

$$\mathcal{F}(e^{-a|t|}) = \frac{2a}{a^2 + 4\pi^2 s^2}$$

Fourier Transform Theorems

Linearity: $\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha F(s) + \beta G(s)$

Stretch: $\mathcal{F}\{g(ax)\} = \frac{1}{|a|} G(\frac{s}{a})$

Shift: $\mathcal{F}\{g(x-a)\} = e^{-i2\pi as} G(s)$

Shift & stretch: $\mathcal{F}\{g(ax-b)\} = \frac{1}{|a|} e^{-i2\pi sb/a} G(\frac{s}{a})$

Rayleigh (Parseval):

$$\begin{aligned} \int_{-\infty}^{\infty} |g(x)|^2 dx &= \int_{-\infty}^{\infty} |G(s)|^2 ds \\ \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx &= \int_{-\infty}^{\infty} F(s) \overline{G(s)} ds \end{aligned}$$

Modulation:

$$\mathcal{F}\{g(x) \cos(2\pi s_0 x)\} = \frac{1}{2} [G(s-s_0) + G(s+s_0)]$$

Autocorrelation: $\mathcal{F}\{\bar{g} \star g\} = |G(s)|^2$

Cross Correlation: $\mathcal{F}\{\bar{g} \star f\} = \overline{G(s)} F(s)$

Derivative:

- $\mathcal{F}\{g'(x)\} = 2\pi is G(s)$

- $\mathcal{F}\{g^{(n)}(x)\} = (2\pi is)^n G(s)$

- $\mathcal{F}\{x^n g(x)\} = (\frac{i}{2\pi})^n G^{(n)}(s)$

Moments:

$$\int_{-\infty}^{\infty} f(x) dx = F(0)$$

$$\int_{-\infty}^{\infty} x f(x) dx = \frac{i}{2\pi} F'(0)$$

$$\int_{-\infty}^{\infty} x^n f(x) dx = (\frac{i}{2\pi})^n F^{(n)}(0)$$

Miscellaneous:

$$\mathcal{F}\left\{\int_{-\infty}^x g(\xi) d\xi\right\} = \frac{1}{2} G(0) \delta(s) + \frac{G(s)}{i2\pi s}$$

The Delta Function: $\delta(x)$

Scaling: $\delta(ax) = \frac{1}{|a|} \delta(x)$

Sifting: $\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a)$

$$\int_{-\infty}^{\infty} \delta(x) f(x+a) dx = f(a)$$

Convolution: $\delta(x) \star f(x) = f(x)$

$$\delta(x-a) \star f(x) = f(x-a)$$

$$\delta(x-a) \star \delta(x-b) = \delta(x-(a+b))$$

Product: $h(x) \delta(x) = h(0) \delta(x)$

Fourier Transform:

$\mathcal{F}\delta = 1$ The complex Fourier series representation:

$$\mathcal{F}(\delta(x - a)) = e^{-2\pi i s a}$$

$$p(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{2\pi i \frac{n}{L} x}$$

Derivatives:

$$\begin{aligned} - & \int_{-\infty}^{\infty} \delta^{(n)}(x) f(x) dx = (-1)^n f^{(n)}(0) \\ - & \delta'(x) * f(x) = f'(x) \\ - & x\delta(x) = 0 \\ - & x\delta'(x) = -\delta(x) \end{aligned}$$

where

$$\begin{aligned} \alpha_n &= \frac{1}{L} F\left(\frac{n}{L}\right) \\ &= \frac{1}{L} \int_{-L/2}^{L/2} p(x) e^{-2\pi i \frac{n}{L} x} dx \end{aligned}$$

Fourier transform of cosine and sine

$$\mathcal{F} \cos 2\pi a t = \frac{1}{2} (\delta(s - a) + \delta(s + a))$$

$$\mathcal{F} \sin 2\pi a t = \frac{1}{2i} (\delta(s - a) - \delta(s + a))$$

Unit step and sgn

$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases} \quad \mathcal{F}H(s) = \frac{1}{2} \left(\delta(s) + \frac{1}{\pi i s} \right)$$

$$\text{sgn } t = \begin{cases} -1, & t < 0 \\ 1, & t > 0 \end{cases} \quad \mathcal{F}\text{sgn}(s) = \frac{1}{\pi i s}$$

The Shah Function: $\text{III}(x)$

$$\text{III}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n), \quad \text{III}_p(x) = \sum_{n=-\infty}^{\infty} \delta(x - np)$$

Sampling: $\text{III}(x)g(x) = \sum_{n=-\infty}^{\infty} g(n)\delta(x - n)$

Periodization: $\text{III}(x) * g(x) = \sum_{n=-\infty}^{\infty} g(x - n)$

Scaling: $\text{III}(ax) = \frac{1}{a} \text{III}_{1/a}(x), \quad a > 0$

Fourier Transform: $\mathcal{F}\text{III} = \text{III}, \quad \mathcal{F}\text{III}_p = \frac{1}{p} \text{III}_{1/p}$

Sampling Theory For a bandlimited function $g(x)$ with $\mathcal{F}g(s) = 0$ for $|s| \geq p/2$

$$\mathcal{F}g = \Pi_p(\mathcal{F}g * \text{III}_p)$$

$$g(t) = \sum_{k=-\infty}^{\infty} g(t_k) \text{sinc}(p(x - t_k)) \quad t_k = k/p$$

Fourier Transforms for Periodic Functions

For a function $p(x)$ with period L , let $f(x) = p(x)\Pi(\frac{x}{L})$. Then

$$p(x) = f(x) * \sum_{n=-\infty}^{\infty} \delta(x - nL)$$

$$P(s) = \frac{1}{L} \sum_{n=-\infty}^{\infty} F\left(\frac{n}{L}\right) \delta\left(s - \frac{n}{L}\right)$$

Linear Systems Let L be a linear system, $w(t) = Lv(t)$, with impulse response $h(t, \tau) = L\delta(t - \tau)$.

Superposition integral:

$$w(t) = \int_{-\infty}^{\infty} v(\tau) h(t, \tau) d\tau$$

A system is time-invariant if:

$$w(t - \tau) = L[v(t - \tau)]$$

In this case $L(\delta(t - \tau)) = h(t - \tau)$ and L acts by convolution:

$$\begin{aligned} w(t) &= Lv(t) = \int_{-\infty}^{\infty} v(\tau) h(t - \tau) d\tau \\ &= (v * h)(t) \end{aligned}$$

The transfer function is the Fourier transform of the impulse response, $H = \mathcal{F}h$. The eigenfunctions of any linear time-invariant system are $e^{2\pi i \nu t}$, with eigenvalue $H(\nu)$:

$$Le^{2\pi i \nu t} = H(\nu) e^{2\pi i \nu t}$$

The Discrete Fourier Transform

N th root of unity:

Let $\omega = e^{2\pi i/N}$. Then $\omega^N = 1$ and the N powers $\underline{1} = \omega^0, \omega, \omega^2, \dots, \omega^{N-1}$ are distinct and evenly spaced along the unit circle.

Vector complex exponentials:

$$\underline{1} = (1, 1, \dots, 1)$$

$$\underline{\omega} = (1, \omega, \omega^2, \dots, \omega^{N-1})$$

$$\underline{\omega}^k = (1, \omega^k, \omega^{2k}, \dots, \omega^{(N-1)k})$$

Cyclic property

$$\underline{\omega}^N = \underline{1} \quad \text{and} \quad \underline{1}, \underline{\omega}, \underline{\omega}^2, \dots, \underline{\omega}^{N-1} \quad \text{are distinct}$$

The vector complex exponentials are orthogonal:

$$\underline{\omega}^k \cdot \underline{\omega}^\ell = \begin{cases} 0, & k \not\equiv \ell \pmod{N} \\ N, & k \equiv \ell \pmod{N} \end{cases}$$

The DFT of order N accepts an N -tuple as input and returns an N -tuple as output. Write an N -tuple as $\underline{f} = (\underline{f}[0], \underline{f}[1], \dots, \underline{f}[N-1])$.

$$\underline{\mathcal{F}}\underline{f} = \sum_{k=0}^{N-1} \underline{f}[k] \underline{\omega}^{-k}$$

Inverse DFT:

$$\underline{\mathcal{F}}^{-1}\underline{f} = \frac{1}{N} \sum_{k=0}^{N-1} \underline{f}[k] \underline{\omega}^k$$

Periodicity of inputs and outputs: If $\underline{F} = \underline{\mathcal{F}}\underline{f}$ then both \underline{f} and \underline{F} are periodic of period N .

Convolution

$$(\underline{f} * \underline{g})[n] = \sum_{k=0}^{N-1} \underline{f}[k] \underline{g}[n-k]$$

Discrete δ :

$$\underline{\delta}_k[m] = \begin{cases} 1, & m \equiv k \pmod{N} \\ 0, & m \not\equiv k \pmod{N} \end{cases}$$

DFT of the discrete δ $\underline{\mathcal{F}}\underline{\delta}_k = \underline{\omega}^{-k}$

DFT of vector complex exponential $\underline{\mathcal{F}}\underline{\omega}^k = N\underline{\delta}_k$

Reversed signal: $\underline{f}^-[m] = \underline{f}[-m]$ $\underline{\mathcal{F}}\underline{f}^- = (\underline{\mathcal{F}}\underline{f})^-$

DFT Theorems

Linearity: $\underline{\mathcal{F}}\{\alpha\underline{f} + \beta\underline{g}\} = \alpha\underline{\mathcal{F}}\underline{f} + \beta\underline{\mathcal{F}}\underline{g}$

Parseval: $\underline{\mathcal{F}}\underline{f} \cdot \underline{\mathcal{F}}\underline{g} = N(\underline{f} \cdot \underline{g})$

Shift: Let $\tau_p\underline{f}[m] = \underline{f}[m-p]$. Then $\underline{\mathcal{F}}(\tau_p\underline{f}) = \underline{\omega}^{-p}\underline{\mathcal{F}}\underline{f}$

Modulation: $\underline{\mathcal{F}}(\underline{\omega}^p\underline{f}) = \tau_p(\underline{\mathcal{F}}\underline{f})$

Convolution: $\underline{\mathcal{F}}(\underline{f} * \underline{g}) = (\underline{\mathcal{F}}\underline{f})(\underline{\mathcal{F}}\underline{g})$

$$\underline{\mathcal{F}}(\underline{f}\underline{g}) = \frac{1}{N}(\underline{\mathcal{F}}\underline{f} * \underline{\mathcal{F}}\underline{g})$$

The Hilbert Transform The Hilbert Transform of $f(x)$:

$$\mathcal{H}f(x) = -\frac{1}{\pi x} * f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi$$

(Cauchy principal value)

Inverse Hilbert Transform $\mathcal{H}^{-1}f = -\mathcal{H}f$

Impulse response: $-\frac{1}{\pi x}$

Transfer function: $i \operatorname{sgn}(s)$

Causal functions: $g(x)$ is causal if $g(x) = 0$ for $x < 0$. A casual signal Fourier Transform $G(s) = R(s) + iI(s)$, where $I(s) = \mathcal{H}\{R(s)\}$.

Analytic signals: The analytic signal representation of a real-valued function $v(t)$ is given by:

$$\begin{aligned} \mathcal{Z}(t) &= \mathcal{F}^{-1}\{2H(s)V(s)\} \\ &= v(t) - i\mathcal{H}v(t) \end{aligned}$$

Narrow Band Signals: $g(t) = A(t) \cos[2\pi f_0 t + \phi(t)]$

Analytic approx: $z(t) \approx A(t)e^{i[2\pi f_0 t + \phi(t)]}$

Envelope: $|A(t)| = |z(t)|$

Phase: $\arg[z(t)] = 2\pi f_0 t + \phi(t)$

Instantaneous freq: $f_i = f_0 + \frac{1}{2\pi} \frac{d}{dt} \phi(t)$

Higher Dimensional Fourier Transform In n -dimensions:

$$\mathcal{F}f(\underline{\xi}) = \int_{\mathbf{R}^n} e^{-2\pi i \underline{x} \cdot \underline{\xi}} f(\underline{x}) d\underline{x}$$

Inverse Fourier Transform:

$$\mathcal{F}^{-1}f(\underline{x}) = \int_{\mathbf{R}^n} e^{2\pi i \underline{x} \cdot \underline{\xi}} f(\underline{\xi}) d\underline{\xi}$$

In 2-dimensions (in coordinates):

$$\mathcal{F}f(\xi_1, \xi_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(x_1 \xi_1 + x_2 \xi_2)} f(x_1, x_2) dx_1 dx_2$$

The Hankel Transform (zero order):

$$F(\rho) = 2\pi \int_0^{\infty} f(r) J_0(2\pi r \rho) r dr$$

The Inverse Hankel Transform (zero order):

$$f(r) = 2\pi \int_0^{\infty} F(\rho) J_0(2\pi r \rho) \rho d\rho$$

Separable functions: If $f(x_1, x_2) = f(x_1)f(x_2)$ then

$$\mathcal{F}f(\xi_1, \xi_2) = \mathcal{F}f(\xi_1)\mathcal{F}f(\xi_2)$$

Two-dimensional rect:

$$\Pi(x_1, x_2) = \Pi(x_1)\Pi(x_2), \quad \mathcal{F}\Pi(\xi_1, \xi_2) = \operatorname{sinc} \xi_1 \operatorname{sinc} \xi_2$$

Two dimensional Gaussian:

$$g(x_1, x_2) = e^{-\pi(x_1^2 + x_2^2)}, \quad \mathcal{F}g = g$$

Fourier transform theorems

Shift: Let $(\tau_{\underline{b}}f)(\underline{x}) = f(\underline{x} - \underline{b})$. Then

$$\mathcal{F}(\tau_{\underline{b}}f)(\underline{\xi}) = e^{-2\pi i \underline{\xi} \cdot \underline{b}} \mathcal{F}f(\underline{\xi})$$

Stretch theorem (special):

$$\mathcal{F}(f(a_1x_1, a_2, x_2)) = \frac{1}{|a_1||a_2|} \mathcal{F}f\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}\right)$$

Stretch theorem (general): If A is an $n \times n$ invertible matrix then

$$\mathcal{F}(f(A\underline{x})) = \frac{1}{|\det A|} \mathcal{F}f(A^{-\top}\underline{\xi})$$

Stretch and shift:

$$\mathcal{F}(f(A\underline{x} + \underline{b})) = \exp(2\pi i \underline{\xi} \cdot A^{-\top}\underline{b}) \frac{1}{|\det A|} \mathcal{F}f(A^{-\top}\underline{\xi})$$

III's and lattices III for integer lattice

$$\begin{aligned} III_{\mathbf{Z}^2}(\underline{x}) &= \sum_{\underline{n} \in \mathbf{Z}^2} \delta(\underline{x} - \underline{n}) \\ &= \sum_{n_1, n_2 = -\infty}^{\infty} \delta(x_1 - n_1, x_2 - n_2) \\ \mathcal{F}III_{\mathbf{Z}^2} &= III_{\mathbf{Z}^2} \end{aligned}$$

A general lattice \mathcal{L} can be obtained from the integer lattice by $\mathcal{L} = A(\mathbf{Z}^2)$ where A is an invertible matrix.

$$III_{\mathcal{L}}(\underline{x}) = \sum_{\underline{p} \in \mathcal{L}} \delta(\underline{x} - \underline{p}) = \frac{1}{|\det A|} III_{\mathbf{Z}^2}(A^{-1}\underline{x})$$

If $\mathcal{L} = A(\mathbf{Z}^2)$ then the reciprocal lattice is $\mathcal{L}^* = A^{-\top}\mathbf{Z}^2$
Fourier transform of $III_{\mathcal{L}}$:

$$\mathcal{F}III_{\mathcal{L}} = \frac{1}{|\det A|} III_{\mathcal{L}^*}$$

Radon transform and Projection-Slice Theorem:

Let $\mu(x_1, x_2)$ be the density of a two-dimensional region. A line through the region is specified by the angle ϕ of its normal vector to the x_1 -axis, and its directed distance ρ from the origin. The integral along a line through the region is given by the Radon transform of μ :

$$\mathcal{R}\mu(\rho, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x_1, x_2) \delta(\rho - x_1 \cos \phi - x_2 \sin \phi) dx_1 dx_2$$

The one-dimensional Fourier transform of $\mathcal{R}\mu$ with respect to ρ is the two-dimensional Fourier transform of μ :

$$\mathcal{F}_{\rho} \mathcal{R}(\mu)(r, \phi) = \mathcal{F}\mu(\xi_1, \xi_2), \quad \xi_1 = r \cos \phi, \quad \xi_2 = r \sin \phi$$

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