# Surface Reconstruction from Derivatives*

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## Abstract

*Most methods to reconstruct surfaces from their derivatives assume two orthogonal derivatives in perfect registration. We propose an approach to use derivatives in arbitrary directions and to register orthogonal derivatives in case they are not registered. We also develop a method to integrate second derivatives in the reconstruction.*


## 1 Introduction

Several 3-D sensing methods measure surface derivatives. Since most applications require 3-D coordinates, surface depth is to be computed from the measured derivatives. Surface can be reconstructed from its derivatives by integration: The depth along one line can be computed by integration along a straight line in one direction of the measured derivatives, using an arbitrary depth at one end of the line. The surface is fully reconstructed by integrating along the lines of the orthogonal direction, using the depth of the initial line as a starting point. This integration method has no redundancy, and it is very sensitive to errors.

Surface reconstruction from measured derivatives was originally used to estimate phase from phase differences [4, 5, 6, 7, 9]. One of the main vision application is in shape from shading algorithms, where surface slopes are estimated from the image intensity [3].

Fourier based methods [3, 5] reconstruct surfaces using all derivatives in two orthogonal directions, the redundancy increases their stability. We generalize the Fourier methods by using measurements in more than two directions, to increase the redundancy of measurements and the accuracy of reconstruction. The directions of derivatives do not have to be orthogonal.

All reconstruction methods require registration of the measured derivatives. In many cases, however, the sensor moves, and the measurements are not registered accurately. We present a method to register derivatives in two orthogonal directions for a possible translation. We also present a method for reconstruction from second derivatives.

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## 2 Two Orthogonal Directions

In this section we review the approach to compute a surface $s(x,y)$ from two orthogonal measured derivatives $\tilde{s}_x(x,y), \tilde{s}_y(x,y)$. We represent the surface as an $N \times N$ array. Derivatives $s_x(x,y), s_y(x,y)$ of a surface $s(x,y)$ satisfy the integrability constraint [3]:

$$\frac{d^2 s}{dxdy} = \frac{d^2 s}{dydx}. \quad (1)$$

The integrability constraint on the derivatives of a surface assures that the integral of derivatives along any close curve will be zero. As the measured derivatives $\tilde{s}_x(x,y), \tilde{s}_y(x,y)$ have errors caused by inaccuracies and noise, they do not satisfy the integrability constraint and do not describe a surface. A surface $\hat{s}(x,y)$ is then computed, such that its derivatives $\hat{s}_x(x,y), \hat{s}_y(x,y)$ are closest to the measured derivatives, minimizing the distance:

$$E(\hat{s}) = \int_{xy} \left[ |\tilde{s}_x - \hat{s}_x|^2 + |\tilde{s}_y - \hat{s}_y|^2 \right] dxdy. \quad (2)$$

Let $F(a)$ be the Fourier transform of the derivative operator $(-1,1)$: $F(a) = 1 - e^{(\frac{-2i\pi a}{N})}$. It has been shown [3, 4], that the Fourier coefficients of the reconstructed surface $\hat{s}(x,y)$ are:

$$\hat{S}(u,v) = \frac{F(u)\tilde{S}_x(u,v) + F(v)\tilde{S}_y(u,v)}{|F(u)|^2 + |F(v)|^2}, \quad (3)$$

where $\hat{S}(u,v)$, $\tilde{S}_x(u,v)$, $\tilde{S}_y(u,v)$ are the Fourier transforms of $\hat{s}(x,y)$, $\tilde{s}_x(x,y)$, $\tilde{s}_y(x,y)$ respectively.

**Summary:** The reconstruction algorithm consists of the following steps:



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1. Performing FFT on the two arrays of the measured derivatives, $\tilde{s}_x(x,y)$ and $\tilde{s}_y(x,y)$, to get $\tilde{S}_x(x,y)$ and $\tilde{S}_y(x,y)$.
2. Computing $\hat{S}(u,v)$ using Eq.(3).
3. Performing the inverse FFT on $\hat{S}(u,v)$ to get the reconstructed surface $\hat{s}(x,y)$.

### 2.1 Using Second Derivative

The multiplication of $\tilde{S}_x(u,v)$ and $\tilde{S}_y(u,v)$ by $F(v)$ and $F(u)$ is simply taking the $x$ and $y$ derivatives in the Fourier domain, thus Eq.(3) is equivalent to integration of the surface Laplacian by dividing it by $|F(u)|^2 + |F(v)|^2$. Therefore Eq.(3) can be rewritten:

$$\hat{S}(u,v) = \frac{\tilde{S}_{x_x}(u,v) + \tilde{S}_{y_y}(u,v)}{|F(u)|^2 + |F(v)|^2}, \qquad (4)$$

where $\tilde{S}_{x_x}$ and $\tilde{S}_{y_y}$ are the Fourier transform of the second derivatives of the surface, $\tilde{s}_{x_x}$ and $\tilde{s}_{y_y}$, respectively. Eq.(4) is being used because of the ease of its derivation, but it is not optimal in the least-squares sense.

## 3 Derivatives in Arbitrary Directions

Let $\tilde{s}_\alpha(x,y)$ be the measured surface derivatives in direction $\alpha$. Given derivatives in a set of directions $\Omega$, $\{\tilde{s}_\alpha | \alpha \in \Omega\}$, we would like to find the surface $\hat{s}(x,y)$ such that its derivatives $\{\hat{s}_\alpha | \alpha \in \Omega\}$ minimize the distance:

$$E(\hat{s}) = \sum_{\alpha\in\Omega} \int_{xy} |\hat{s}_\alpha(x,y) - \tilde{s}_\alpha(x,y)|^2 dxdy. \qquad (5)$$

As in the case with derivatives along two orthogonal directions, surface reconstruction is done using the Fourier transform. It can be shown that the surface $\hat{s}(x,y)$ that minimizes the error (Eq.(5)) between the measured directional derivatives $\tilde{s}_\alpha(x,y)$, and the directional derivatives of the reconstructed surface $\hat{s}_\alpha(x,y)$, also minimizes the error between the respective Fourier coefficients:

$$E(\hat{s}) = \sum_{\alpha\in\Omega} \sum_{u,v=0}^{N-1} |\hat{S}_\alpha(u,v) - \tilde{S}_\alpha(u,v)|^2. \qquad (6)$$

We will generalize $F_\alpha(u,v)$ to be the Fourier transform of the derivative in direction $\alpha$:

$$F_\alpha(u,v) =$$
$$\cos(\alpha)(1 - exp(\frac{-2i\pi u}{N})) + \sin(\alpha)(1 - exp(\frac{-2i\pi v}{N})).$$

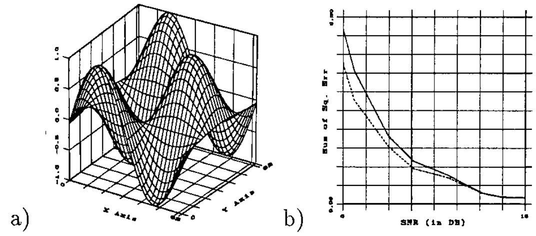


Figure 1: Surface Reconstruction using derivatives in four directions: synthetic data.
a) Reconstructed surface.
b) Error of reconstructed surface as function of noise. Dashed line uses four directions, and solid line uses two directions. Advantage of using four direction is seen at high noise level (Low SNR).

Since: $\hat{S}_\alpha(u,v) = \hat{S}(u,v)F_\alpha(u,v)$, the error $E(\hat{s})$ is then rewritten as:

$$E(\hat{s}) = \sum_{\alpha\in\Omega} \sum_{u,v=0}^{N-1} |\hat{S}(u,v)F_\alpha(u,v) - \tilde{S}_\alpha(u,v)|^2. \qquad (7)$$

Minimization of Eq.(7) provides the Fourier coefficients of the reconstructed surface:

$$\hat{S}(u,v) = \frac{\sum_{\alpha\in\Omega} \tilde{S}_\alpha(u,v) F_\alpha(u,v)}{\sum_{\alpha\in\Omega} |F_\alpha(u,v)|^2}. \qquad (8)$$

Surface reconstruction is therefore done by using the same steps as described in previous section, but having more freedom in the measuring directions. Using more then two derivatives has the advantage of processing more information about the surface, and integrating it with minimum error propagation. In case that each measured directional derivative $\tilde{s}_\alpha$ has a different confidence, say $K_\alpha$, Eq.(8) can be modified, getting:

$$\hat{S}(u,v) = \frac{\sum_{\alpha\in\Omega} K_\alpha \tilde{S}_\alpha(u,v) F_\alpha(u,v)}{\sum_{\alpha\in\Omega} K_\alpha |F_\alpha(u,v)|^2}. \qquad (9)$$

## 4 Cyclic Basis Functions

The Fourier basis functions generate only cyclic surfaces. To make the measured derivatives $\tilde{s}_x(x,y)$, and $\tilde{s}_y(x,y)$ in an $N \times N$ arrays, satisfy the cyclic property, we generate an additional last column to the $x$ derivatives and an additional last row to the $y$ derivatives. The added values are [5]:

$$\tilde{s}_x(N,y) = -\sum_{x=0}^{N-1} \tilde{s}_x(x,y),$$

$$\tilde{s}_y(x, N) = -\sum_{y=0}^{N-1} \tilde{s}_y(x, y). \qquad (10)$$

Consider the case of two directional derivatives measured in arbitrary directions, $s_\alpha(x, y)$ and $s_\beta(x, y)$, which are organized in $N \times N$ arrays. We will make them satisfy the cyclic property by adding one row and one column. Since:

$$s_\alpha(x, y) = \cos(\alpha) s_x(x, y) + \sin(\alpha) s_y(x, y), \qquad (11)$$

it holds for every row of the directional derivatives that:

$$\sum_{x=0}^{N-1} \tilde{s}_\alpha(x, y) = \cos(\alpha) \sum_{x=0}^{N-1} \tilde{s}_x(x, y) + \sin(\alpha) \sum_{x=0}^{N-1} \tilde{s}_y(x, y)$$
$$\sum_{x=0}^{N-1} \tilde{s}_\beta(x, y) = \cos(\beta) \sum_{x=0}^{N-1} \tilde{s}_x(x, y) + \sin(\beta) \sum_{x=0}^{N-1} \tilde{s}_y(x, y). \qquad (12)$$

The two unknowns in these equations are $\sum_{x=0}^{N-1} \tilde{s}_x$ and $\sum_{x=0}^{N-1} \tilde{s}_y$, which can be computed by solving these two equations. Applying Eq.(11) on $\tilde{s}_\alpha(N, y)$, and substituting Eq.(10) in the result to satisfy $x$ cyclic surface, leads to:

$$\tilde{s}_\alpha(N, y) = -\cos(\alpha) \sum_{x=0}^{N-1} \tilde{s}_x(x, y) + \sin(\alpha) \tilde{s}_y(N, y). \qquad (13)$$

The above constraint restricts the $x$ component of $\tilde{s}_\alpha(N, y)$ to be the sum of the $x$ components of $\tilde{s}_\alpha(x, y)$, $0 \leq x \leq N-1$. For determining $\tilde{s}_y(N, y)$ we apply the integrability constraint on the cyclic surface (Fig.2):

$$\tilde{s}_y(N, y) = \tilde{s}_x(N, y) - \tilde{s}_x(N, y+1) + \tilde{s}_y(0, y), \qquad (14)$$

where $\tilde{s}_x(N, y)$, $\tilde{s}_x(N, y+1)$ are obtained by using Eq.(12), and $\tilde{s}_y(0, y)$ can be obtained from $\tilde{s}_\alpha(0, y)$ and $\tilde{s}_\beta(0, y)$.

**Summary:** The values $\tilde{s}_\alpha(N, y)$ at the $(N+1)st$ column are generated by:

- Solving Eq.(12) to obtain $\sum_{x=0}^{N-1} \tilde{s}_x$.
- Computing $\tilde{s}_y(0, y)$ from $\tilde{s}_\alpha(0, y)$ and $\tilde{s}_\beta(0, y)$.
- Using the values from the first two steps to calculate $\tilde{s}_y(N, y)$ from Eq.(14).
- Substituting the values $\tilde{s}_y(N, y)$ and $\sum_{x=0}^{N-1} \tilde{s}_x$ in Eq.(13) to get $\tilde{s}_\alpha(N, y)$.

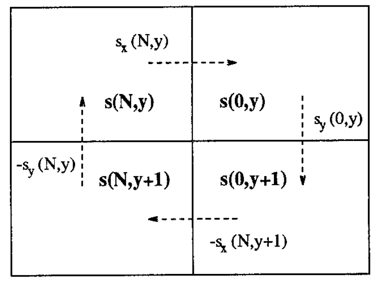



Figure 2: Derivatives in a cyclic surface: $s(N+1, y) = s(0, y)$.

The values of $\tilde{s}_\alpha(x, N)$ can be generated similarly. $\tilde{s}_\alpha(N, N)$ is than computed by:

$$-\tilde{s}_\alpha(N, N) = \cos(\alpha) \sum_{x=0}^{N-1} \tilde{s}_x(x, N) + \sin(\alpha) \sum_{y=0}^{N-1} \tilde{s}_y(N, y). \qquad (15)$$

## 5 Registration of Derivatives

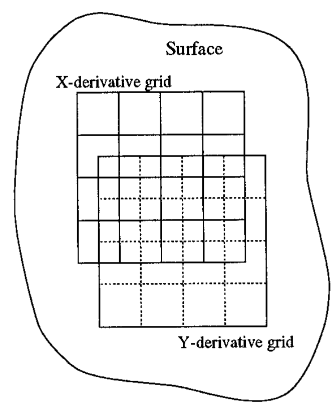



Figure 3: Measurements of derivatives might not be aligned due to translational movement.

At least two measurements of derivatives from different directions are necessary for surface reconstruction. In many cases the derivatives might not be aligned, but alignment is essential for accurate reconstruction. To align the directional derivatives we use the observation that the sum of derivatives of a surface along a closed contour is zero. When derivatives in two directions are not aligned, the sum of derivatives on closed contours will not be zero. The minimal closed curve is a block of $2 \times 2$ pixels, and the sum of the mea-

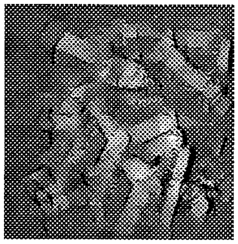
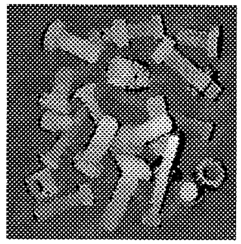
a)  b) 

Figure 4: The effect of non-registered derivatives on surface reconstruction of range images.

a) Reconstruction with non-registered derivatives. Derivatives are displaced by about 21 pixels (picture size is 256 × 256).

b) Reconstruction with registration: The parameters of translation were obtained to align the derivatives before reconstruction.

sured derivatives along this curve given a translation $(t_x, t_y)$ between the measurements is:

$$c(t_x, t_y, x, y) = \tilde{s}_x(x, y) + \tilde{s}_y(x + 1 - t_x, y - t_y) \\ -\tilde{s}_x(x, y + 1) - \tilde{s}_y(x - t_x, y - t_y). \quad (16)$$

Under discrete derivation Eq.(16) can be written as:

$$c(t_x, t_y, x, y) = \tilde{s}_{x_y}(x, y) - \tilde{s}_{y_x}(x - t_x, y - t_y). \quad (17)$$

Minimizing $C(t_x, t_y) = \sum_{x,y}(c(t_x, t_y, x, y)^2)$ over $t_x$ and $t_y$ will be used to register the derivatives. We will do it by applying a motion registration algorithm to find the translation between $\tilde{s}_{x_y}$ and $\tilde{s}_{y_x}$ [1, 2].

Surface reconstruction is performed after the registration of the directional derivatives. The surface is divided into three areas: An area having measured derivatives in both directions, and two areas having measured derivatives in only one direction. The reconstruction algorithm is first applied on the area having measurements in two directions. Reconstruction in the other two area is done by integrating the derivative in one direction starting from the values on the boundary of the first computed region.

## 6 Concluding Remarks

We have proposed an approach to reconstruct a surfaces from derivatives in any arbitrary directions. A method to register derivatives in different directions has been also presented. These steps contribute to the effort of generating surfaces from measured directional derivatives, overcoming the limitation of existing methods that only work on registered orthogonal derivatives.

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