

# Fourier Transforms

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## 1 Discrete vs. continuous FT

continuous Fourier transform

$$\hat{f}(k) = \int_{-\pi}^{\pi} f(x) \cdot e^{-2\pi i k x} dx \quad (1)$$

discrete Fourier transform

$$\hat{f}_k = \frac{1}{d} \sum_{n=0}^{N-1} f\left(x = \frac{n}{N}d\right) \cdot e^{(-2\pi i \cdot \frac{k}{d} \cdot \frac{n}{N} \cdot d)} \cdot \frac{d}{N} \quad (2)$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} f\left(x = \frac{n}{N}d\right) \cdot e^{-\frac{2\pi i}{N} \cdot k \cdot n} \quad (3)$$

## 2 Differentiation

Expression in frequency and angular frequency

$$\hat{f}(k) = \frac{1}{d} \int_0^d f(x) e^{\frac{-2\pi i}{d} k x} dx \quad (4)$$

$$= \frac{1}{d} \int_0^d f(x) e^{-2\pi i \xi x} dx \quad (5)$$

$$\hat{f}' = \frac{\partial}{\partial x} \left( \int_{-\infty}^{\infty} \hat{f}(x) \cdot e^{i\xi x} dk \right) \quad (6)$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left( \hat{f}(x) \cdot e^{i\xi x} \right) dk \quad (7)$$

$$= \int_{-\infty}^{\infty} i\xi \cdot \hat{f}(x) \cdot e^{i\xi x} dk \quad (8)$$

### 3 Transform

example with  $N = 4$  and  $x = \sin\left(\frac{n}{N} \cdot 2\pi\right)$

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{2\pi i}{N} \cdot k \cdot n}; \quad k = 0; 1; \dots; N-1 \quad (9)$$

$$= \sum_{n=0}^{N-1} x_n \cdot \left( \cos\left(-\frac{2\pi}{N} \cdot k \cdot n\right) + i \cdot \sin\left(-\frac{2\pi}{N} \cdot k \cdot n\right) \right) \quad (10)$$

$$X_0 = \sum_{n=0}^{N-1} x_n e \quad (11)$$

$$= 0 + 1 + 0 + (-1) = 0 \quad (12)$$

$$X_1 = \sum_{n=0}^{N-1} x_n e^{-i\frac{2\pi}{N} \cdot 1 \cdot n} = 0 + e^{-i\frac{2\pi}{N} \cdot 1 \cdot 1} + 0 + e^{-i\frac{2\pi}{N} \cdot 1 \cdot 3} \quad (13)$$

$$= 0 + (-2i) \quad (14)$$

$$X_2 = 0 + 0i \quad (15)$$

$$X_3 = 0 + 2i \quad (16)$$

$X_2$  is Nyquist frequency and has only a real part,  $X_3$  is conjugate complex of  $X_1$  for real only input.

## 4 Inverse Transform

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{\frac{2\pi i}{N} \cdot k \cdot n} \quad (17)$$

$$x_0 = \frac{1}{4} (0 - 2ie^0 + 0 + 2ie^0) = 0 \quad (18)$$

$$x_1 = \frac{1}{4} \left( 0 - 2ie^{\frac{2\pi i}{4} \cdot 1 \cdot 1} + 0 + 2ie^{\frac{2\pi i}{4} \cdot 3 \cdot 1} \right) = 1 \quad (19)$$

$$x_2 = 0 \quad (20)$$

$$x_3 = -1 \quad (21)$$